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Concurrence vectors in arbitrary multipartite quantum systems

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Abstract

For a given pure state of a multipartite system, the concurrence vector is defined by employing the defining representation of generators of the corresponding rotation groups. The norm of a concurrence vector is considered as a measure of entanglement. For the multipartite pure state, the concurrence vector is regarded as the direct sum of concurrence subvectors in the sense that each subvector is associated with a pair of particles. It is proposed to use the norm of each subvector as the contribution of the corresponding pair in entanglement of the system.

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1. Introduction

Quantum entanglement, as the most intriguing feature of quantum mechanics, has been investigated for decades in relation to quantum nonseparability and the violation of Bell's inequality [1–3]. In the last decade, it has been regarded as a valuable resource for quantum communications and information processing [4–6], so, as with other resources such as free energy and information, quantification of entanglement is necessary to understand and develop the theory.

From the various measures proposed to quantify entanglement, the entanglement of formation has been widely accepted which in fact intends to quantify the resources needed to create a given entangled state [6]. In the case of a pure state, if the density matrix obtained from the partial trace over other subsystems is not pure the state is entangled. Consequently, for the pure state $|\psi\rangle$ of a bipartite system, entropy of the density matrix associated with either of the two subsystems is a good measure of entanglement,

$$E(\psi) = -\mathrm{Tr}(\rho_A \log_2 \rho_A) = -\mathrm{Tr}(\rho_B \log_2 \rho_B), \qquad (1)$$

where $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ and ρ_B is defined similarly. Due to classical correlations existing in the mixed state each subsystem can have a non-zero entropy even if there is no entanglement, therefore the von Neumann entropy of a subsystem is no longer a good measure of entanglement. For a mixed state, the entanglement of formation (EoF) is defined as the minimum of an average entropy of the state over all pure state decompositions of the state [6],

$$E_f(\rho) = \min \sum_i p_i E(\psi_i). \tag{2}$$

Although the entanglement of formation has most widely been accepted as an entanglement measure, there is no known explicit formula for the EoF of a general state of bipartite systems except for $2 \otimes 2$ quantum systems [7] and special types of mixed states with definite symmetry such as isotropic states [8] and Werner states [9]. Remarkably, Wootters has shown that EoF of a two-qubit mixed state ρ is related to a quantity called concurrence as [7]

$$E_f(\rho) = H\big(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 - C^2}\big), \tag{3}$$

where $H(x) = -x \ln x - (1 - x) \ln(1 - x)$ is the binary entropy and concurrence $C(\rho)$ is defined by

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{4}$$

where the λ_i are the non-negative eigenvalues, in decreasing order, of the Hermitian matrix $R \equiv \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$ and

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y), \tag{5}$$

where ρ^* is the complex conjugate of ρ when it is expressed in a standard basis such as $\{|11\rangle, |12\rangle, |21\rangle, |22\rangle\}$ and σ_y represents the Pauli matrix in a local basis $\{|1\rangle, |2\rangle\}$. Furthermore, the EoF is a monotonically increasing function of the concurrence $C(\rho)$, so one can use concurrence directly as a measure of entanglement. For pure state $|\psi\rangle = a_{11}|11\rangle + a_{12}|12\rangle + a_{21}|21\rangle + a_{22}|22\rangle$, the concurrence takes the form

$$C(\psi) = |\langle\psi|\tilde{\psi}\rangle| = 2|a_{11}a_{22} - a_{12}a_{21}|. \tag{6}$$

Because of the relation between concurrence and entanglement of formation it is, therefore, interesting to ask whether concurrence can be generalized to larger quantum systems. Indeed attempts have been made to generalize the definition of concurrence to higher dimensional composite systems [10–17]. Uhlmann generalized the concept of concurrence by considering arbitrary conjugations acting on arbitrary Hilbert spaces [10]. His motivation is based on the fact that the tilde operation on a pair of qubits is an example of conjugation, that is, an anti-unitary operator whose square is the identity. Rungta *et al* defined the so-called I-concurrence in terms of a universal inverter which is a generalization to higher dimensions of a two-qubit spin flip operation, therefore, the pure state concurrence in arbitrary dimensions takes the form [11]

$$C(\psi) = \sqrt{\langle\psi|\mathcal{S}_{N_1} \otimes \mathcal{S}_{N_2}(|\psi\rangle\langle\psi|)|\psi\rangle} = \sqrt{2\big(1 - \mathrm{Tr}\big(\rho_A^2\big)\big)}. \tag{7}$$

Another generalization is proposed by Audenaert *et al* [12] by defining a concurrence vector in terms of a specific set of antilinear operators. As pointed out by Wootters, it turns out that the length of the concurrence vector is equal to the definition given in equation (7) [18]. Albeverio and Fei also generalized the notion of concurrence by using invariants of local unitary transformations as [13]

$$C(\psi) = \sqrt{\frac{N}{N-1}\big(I_0^2 - I_1^2\big)} = \sqrt{\frac{N}{N-1}\big(1 - \mathrm{Tr}\big(\rho_A^2\big)\big)}, \tag{8}$$

which turns out to be the same as that of Rungta *et al* up to a whole factor. In equation (8) I_0 and I_1 are two former invariants of the group of local unitary transformations. As a complete characterization of entanglement of a bipartite state in arbitrary dimensions may require a quantity which, even for pure states, does not reduce to a single number [19–23], Fan *et al* defined the concept of a concurrence hierarchy as $N-1$ invariants of a group of local unitary for N-level systems [14]. Badziag *et al* [15] also introduced a multi-dimensional generalization of concurrence. Recently, Li *et al* used a fundamental representation of the A_N Lie algebra and proposed concurrence vectors for a bipartite system of arbitrary dimension as [16]

$$\mathbf{C} = \{\langle\psi|(E_\alpha - E_{-\alpha}) \otimes (E_\beta - E_{-\beta})|\psi^*\rangle|\alpha, \beta \in \Delta^+\}, \tag{9}$$

where Δ^+ denotes the set of positive roots of the A_{N-1} Lie algebra. An extension of the notion of Wooters concurrence to multi-qubit systems is also proposed in [17].

In this contribution, I generalize the notion of concurrence vectors to arbitrary multipartite systems. The motivation is based on the fact that Wootters concurrence of a pair of qubits can be obtained by defining the tilde operation as $|\tilde{\psi}\rangle = S\otimes S|\psi^*\rangle$ instead of $|\tilde{\psi}\rangle = \sigma_y\otimes\sigma_y|\psi^*\rangle$, here S is the only generator of rotation group $SO(2)$ in such a basis that $(S)_{ij} = \epsilon_{ij}$ where $\epsilon_{12} = -\epsilon_{21} = 1$ and $\epsilon_{11} = \epsilon_{22} = 0$. Therefore, a natural generalization of the spin flip operation for arbitrary bipartite systems leads to a vector whose components are obtained by employing the tensor product of generators of the corresponding rotation groups. A suitable generalization of the definition for the multipartite system is also proposed by defining the concurrence vector as a direct sum of concurrence subvectors in the sense that each subvector corresponds to one pair of particles. Therefore, it is proposed to use the norm of each subvector as a measure of entanglement shared between the corresponding pair of particles. A criterion for the separability of bipartite states is then given as: a state is separable if and only if the norm of its concurrence vector vanishes. For multipartite systems, the vanishing of the concurrence vectors is a necessary but not a sufficient condition for separability.

The paper is organized as follows: In section 2, the definition of concurrence vectors is given. In section 3, the generalization of the concurrence vector for a multipartite system is proposed. Some examples are also considered in this section. The paper is concluded in section 4 with a brief conclusion.

2. Concurrence vectors for bipartite pure states

In this section, we give a generalization of the concurrence for an arbitrary bipartite pure state. For motivation, let us first consider a pure state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^3$ with the following generic form

$$|\psi\rangle = \sum_{i=1}^{2}\sum_{j=1}^{3} a_{ij}|e_i \otimes e_j\rangle, \tag{10}$$

where $|e_i\rangle (i = 1, 2)$ and $|e_j\rangle (j = 1, 2, 3)$ are orthonormal real bases of Hilbert space $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively. Of course, by means of the Schmidt decomposition one can consider $|\psi\rangle$ as a vector in a $\mathbb{C}^2 \otimes \mathbb{C}^2$ Hilbert space, but to see the main idea of the paper we do not use the Schmidt decomposition. It can be easily seen that the entanglement of $|\psi\rangle$ can be written as $E_f(\psi) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-C^2}\right)$, where concurrence C is defined by

$$C = 2\sqrt{|a_{12}a_{23} - a_{13}a_{22}|^2 + |a_{11}a_{23} - a_{13}a_{21}|^2 + |a_{11}a_{22} - a_{12}a_{21}|^2}. \tag{11}$$

On the other hand, equation (11) can also be written as

$$C = \sqrt{\sum_{\alpha=1}^{3} |\langle\psi|\tilde{\psi}_\alpha\rangle|^2}, \tag{12}$$

where $|\tilde{\psi}_\alpha\rangle$ are defined by

$$|\tilde{\psi}_\alpha\rangle = (S \otimes L_\alpha)|\psi^*\rangle, \tag{13}$$

where S is the only generator of two-dimensional rotation group $SO(2)$ with matrix elements $(S)_{ij} = \epsilon_{ij}$ and L_α with matrix elements $(L_\alpha)_{jk} = \epsilon_{\alpha jk}$ denotes three generators of an $SO(3)$ group. Here ϵ_{ij} is defined by $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$ and $\epsilon_{\alpha jk}$ is antisymmetric under the interchange of any two indices and $\epsilon_{123} = 1$.

Similarly, for a pure state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^N$ with the generic form

$$|\psi\rangle = \sum_{i=1}^{2}\sum_{j=1}^{N} a_{ij}|e_i \otimes e_j\rangle, \tag{14}$$

entanglement $E(\psi)$ is obtained by equation (3) with the following:

$$C = 2\sqrt{\sum_{j<k}^{N} |a_{1j}a_{2k} - a_{2j}a_{1k}|^2}. \tag{15}$$

It is straightforward to see that equation (15) can be expressed as

$$C = \sqrt{\sum_{\alpha=1}^{N(N-1)/2} |\langle\psi|\tilde{\psi}_\alpha\rangle|^2}. \tag{16}$$

Here $|\tilde{\psi}_\alpha\rangle = (S \otimes L_\alpha)|\psi^*\rangle$, $\alpha = 1, \ldots, N(N-1)/2$, where L_α are generators of an $SO(N)$ group with matrix elements $(L_\alpha)_{kl} = (L_{[j_1 j_2\cdots j_{N-2}]})_{kl} = \epsilon_{[j_1 j_2\cdots J_{N-2}]kl}$ where α is used to denote the set of $N-2$ indices $[j_1 j_2 \cdots j_{N-2}]$ with $1 \leqslant j_1 < j_2 < \cdots < j_{N-2} \leqslant N$ in order to label $N(N-1)/2$ generators of $SO(N)$, and $\epsilon_{j_1 j_2\cdots j_N}$ is antisymmetric under the interchange of any two indices with $\epsilon_{12\cdots N} = 1$. To achieve equation (15) from equation (16) we used the following equations:

$$\epsilon_{kl}\epsilon_{k'l'} = \delta_{kk'}\delta_{ll'} - \delta_{kl'}\delta_{k'l}, \tag{17}$$

$$\sum_{1\leqslant j_1<j_2<\cdots<j_{N-2}\leqslant N} \epsilon_{[j_1 j_2\cdots j_{N-2}]kl}\epsilon_{[j_1 j_2\cdots j_{N-2}]k'l'} = \delta_{kk'}\delta_{ll'} - \delta_{kl'}\delta_{k'l}. \tag{18}$$

Next, to generalize the above definition of concurrence for an arbitrary bipartite pure state let $|\psi\rangle$ be a pure state in Hilbert space $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ with the following decomposition

$$|\psi\rangle = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} a_{ij}|e_i \otimes e_j\rangle. \tag{19}$$

Now we define the concurrence vector **C** with components $C_{\alpha\beta}$ as

$$C_{\alpha\beta} = \langle\psi|\tilde{\psi}_{\alpha\beta}\rangle, \qquad |\tilde{\psi}_{\alpha\beta}\rangle = (L_\alpha \otimes L_\beta)|\psi^*\rangle. \tag{20}$$

where L_α, $\alpha = 1, \ldots, N_1(N_1-1)/2$ and L_β, $\beta = 1, \ldots, N_2(N_2-1)/2$ are generators of $SO(N_1)$ and $SO(N_2)$ respectively. Now the norm of the concurrence vector can be defined as a measure of entanglement, i.e.,

$$C = |\mathbf{C}| = \sqrt{\sum_{\alpha=1}^{N_1(N_1-1)/2}\sum_{\beta=1}^{N_2(N_2-1)/2} |C_{\alpha\beta}|^2}. \tag{21}$$

By using equation (18) we can evaluate the concurrence in terms of parameters a_{ij} where we get

$$C = 2\sqrt{\sum_{i<j}^{N_1}\sum_{k<l}^{N_2}|a_{ik}a_{jl} - a_{il}a_{jk}|^2}. \tag{22}$$

It is clear that $C(\psi)$ is zero when $|\psi\rangle$ is factorizable, i.e., $a_{ij} = b_i c_j$ for some $b_i, c_j \in \mathbb{C}$. On the other hand, C takes its maximum value $\sqrt{2(N-1)/N}$ with $N = \min(N_1, N_2)$, when $|\psi\rangle$ is a maximally entangled state. It should be noted that the result is the same as that obtained in [13], up to a whole factor, therefore it is also in accordance with the result obtained from the definition given in [11]. It should also be mentioned that the result is equal to the bipartite pure state concurrence that Badsiag *et al* have defined in terms of a trace norm of the concurrence matrix [15]. Interestingly, components of the concurrence vector given in equation (20) are the same as the elements of the concurrence matrix of [15]. As a matter of fact, the definition given in equation (20) for concurrence vectors is closely related to the definition proposed in [16]. Actually, all bipartite generalizations of the concurrence lead to equation (22). However, our objective here is to generalize the definition for multipartite systems.

3. Concurrence vectors for multipartite systems

In order to further generalize the concept of a concurrence vector to multipartite systems, let us first analyse the problem that arises in the definition of the pairwise entanglement between the particles. In equation (20) $|\psi^*\rangle$ can also be written as $|\psi^*\rangle = (K_1 \otimes K_2)|\psi\rangle$ where K_1 and K_2 are the complex conjugation operators acting in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ respectively. Although the action of the direct product of two anti-unitary transformations $K_1 \otimes K_2$ on a general ket $|\psi\rangle \in \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ can be properly defined, the combination of an anti-unitary and a unitary transformation such as $K_1 \otimes K_2 \otimes I_3$ cannot be properly defined on a general ket $|\psi\rangle \in \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2} \otimes \mathbb{C}^{N_3}$ except that $|\psi\rangle$ is factorized as $|\psi\rangle = |\psi_{12}\rangle \otimes |\psi_3\rangle$ where $|\psi_{12}\rangle \in \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ and $|\psi_3\rangle \in \mathbb{C}^{N_3}$. This ambiguity can be removed in the Hilbert Schmidt basis [24] of the corresponding system with

$$\rho^{T_{12}} = (K_1 \otimes K_2 \otimes I_3)\rho(K_1 \otimes K_2 \otimes I_3), \tag{23}$$

for any $\rho = |\psi\rangle\langle\psi|$, whether $|\psi\rangle$ is factorizable or not. In equation (23) $\rho^{T_{12}}$ is the partial transpose of ρ with respect to particles 1 and 2.

Now to generalize the concept of a concurrence vector to multipartite systems, let us consider an m-partite pure state $|\psi\rangle \in \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2} \otimes \cdots \otimes \mathbb{C}^{N_m}$ which in the standard basis has the following decomposition:

$$|\psi\rangle = \sum_{i_1}^{N_1}\sum_{i_2}^{N_2}\cdots\sum_{i_m}^{N_m} a_{i_1 i_2 \cdots i_m}\big|e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_m}\big\rangle. \tag{24}$$

We first define the pairwise entanglement between the particles. Let $\rho = |\psi\rangle\langle\psi|$ be the density matrix corresponding to pure state (24). With $\rho^{T_{ij}}$, we denote the matrix obtained from ρ by partial transposition with respect to subsystems i and j, i.e.,

$$\rho^{T_{ij}} = (|\psi\rangle\langle\psi|)^{T_{ij}}. \tag{25}$$

Next, we define $\Pi_{i=1}^m N_i(N_i - 1)/2$ dimensional concurrence vector $\mathbf{C}$ with components $C^{\{ij\}}_{\alpha_i\alpha_j}$ as

$$C^{\{ij\}}_{\alpha_i\alpha_j} = \sqrt{\langle\psi|\tilde{\rho}^{\{ij\}}_{\alpha_i,\alpha_j}|\psi\rangle}, \tag{26}$$

where

$$\tilde{\rho}^{\{ij\}}_{\alpha_i,\alpha_j} = M^{\{ij\}}_{\alpha_i\alpha_j}\rho^{T_{ij}}M^{\{ij\}}_{\alpha_i\alpha_j} \tag{27}$$

with

$$M^{\{ij\}}_{\alpha_i\alpha_j} = I_1\otimes\cdots\otimes I_{i-1}\otimes L_{\alpha_i}\otimes I_{i+1}\otimes\cdots\otimes I_{j-1}\otimes L_{\alpha_j}\otimes I_{j+1}\otimes\cdots\otimes I_m, \tag{28}$$

for $1 \leqslant i < j \leqslant m, \alpha_i = 1,\ldots,N_i(N_i-1)/2$ and $\alpha_j = 1,\ldots,N_j(N_j-1)/2$. Here I_k denotes the identity matrix in Hilbert space of particle k, and L_{α_i} represents the set of $N_i(N_i-1)/2$ generators of an $SO(N_i)$ group with the following matrix elements,

$$\langle k_i|L_{\alpha_i}|l_i\rangle = \left(L_{\alpha_i}\right)_{k_il_i} = \left(L_{[i_1i_2\cdots i_{N_i-2}]}\right)_{k_il_i} = \epsilon_{[i_1i_2\cdots i_{N_i-2}]k_il_i}, 1\leqslant i_1 < i_2 < \cdots < i_{N_i-2}\leqslant N_i, \tag{29}$$

and L_{α_j} are generators of an $SO(N_j)$ group with a similar definition. The concurrence vector $\mathbf{C}$ is defined in such a way that it involves all two-level entanglements shared between all pairs of particles. Moreover, we can consider vector $\mathbf{C}$ as a direct sum of elementary subvectors $\mathbf{C}^{\{ij\}}$, i.e.,

$$\mathbf{C} = \sum_{\oplus ij}\mathbf{C}^{\{ij\}}, \tag{30}$$

such that each subvector $\mathbf{C}^{\{ij\}}$ corresponds to pair i and j of particles. Accordingly, the entanglement contribution of pair i and j in the entanglement of $|\psi\rangle$ can be defined as the norm of the concurrence subvector $\mathbf{C}^{\{ij\}}$, that is

$$\begin{aligned} C^{\{ij\}} = |\mathbf{C}^{\{ij\}}| &= \sqrt{\sum_{\alpha_i=1}^{N_i(N_i-1)/2}\sum_{\alpha_j=1}^{N_j(N_j-1)/2}\langle\psi|\tilde{\rho}^{\{ij\}}_{\alpha_i,\alpha_j}|\psi\rangle} \\ &= \Bigg\{\sum_{\{K\}}\sum_{\{L\}}\sum_{k_i<l_i}^{N_i}\sum_{k_j<l_j}^{N_j}|a_{\{k_i,k_j,K\}}a_{\{l_i,l_j,L\}} - a_{\{k_i,l_j,K\}}a_{\{l_i,k_j,L\}} \\ &\quad - a_{\{l_i,k_j,K\}}a_{\{k_i,l_j,L\}} + a_{\{l_i,l_j,K\}}a_{\{k_i,k_j,L\}}|^2\Bigg\}^{1/2}, \end{aligned} \tag{31}$$

where in the last line we used the following equations:

$$\begin{aligned} &\sum_{[\alpha_i]}\epsilon_{[\alpha_i]k_il_i}\epsilon_{[\alpha_i]k_i'l_i'} = \delta_{k_ik_i'}\delta_{l_il_i'} - \delta_{k_il_i'}\delta_{k_i'l_i}, \\ &\sum_{[\alpha_j]}\epsilon_{[\alpha_j]k_jl_j}\epsilon_{[\alpha_j]k_j'l_j'} = \delta_{k_jk_j'}\delta_{l_jl_j'} - \delta_{k_jl_j'}\delta_{k_j'l_j}. \end{aligned} \tag{32}$$

In equation (31), $\{k_i, k_j, K\}$ stands for m indices such that k_i and k_j correspond to subsystems i and j respectively, and K denotes the set of $m-2$ indices for other subsystems. Also $\sum_{\{K\}}$ stands for summation over indices of all subsystems except subsystems i and j.

It can be shown that $C^{\{ij\}}$ is zero when $|\psi\rangle$ is factorizable among indices i and the rest of the system, i.e., when there exist some $b_{k_i}, c_{\{k_j,K\}} \in \mathbb{C}$ such that $a_{\{k_i,k_j,K\}} = b_{k_i}c_{\{k_j,K\}}$. A similar statement is also correct when $|\psi\rangle$ is factorizable among indices j and the rest of the system. Also when two subsystems i and j are disentangled from the rest of the system, i.e., $a_{\{k_i,k_j,K\}} = b_{k_i,k_j}c_{\{K\}}$ for some $b_{k_i,k_j}, c_{\{K\}} \in \mathbb{C}$, equation (31) takes the form of equation (22), as we expect. This feature of $C^{\{ij\}}$ shows that it can be considered as the pairwise entanglement between the subsystems i and j.

Finally, the total concurrence of $|\psi\rangle$ may be defined as the norm of the concurrence vector $\mathbf{C}$, i.e.,

$$C = |\mathbf{C}| = \sqrt{\sum_{1\leqslant i<j\leqslant m} |\mathbf{C}^{\{ij\}}|^2}$$
$$= \left\{ \sum_{1\leqslant i<j\leqslant m} \sum_{\{K\}} \sum_{\{L\}} \sum_{k_i<l_i}^{N_i} \sum_{k_j<l_j}^{N_j} |a_{\{k_i,k_j,K\}}a_{\{l_i,l_j,L\}} - a_{\{k_i,l_j,K\}}a_{\{l_i,k_j,L\}} - a_{\{l_i,k_j,K\}}a_{\{k_i,l_j,L\}} + a_{\{l_i,l_j,K\}}a_{\{l_i,l_j,L\}}|^2 \right\}^{1/2}. \tag{33}$$

It is clear that if $|\psi\rangle$ is completely separable, if $a_{i_1i_2\cdots i_m} = a_{i_1}b_{i_2}\cdots$ for some $a_{i_1}, b_{i_2}, \ldots \in \mathbb{C}$, then all $C^{\{ij\}}_{\alpha_i\alpha_j}$ are zero and the entanglement of the state becomes zero. This is, of course, the first condition that any good measure of entanglement should satisfy. Furthermore, the entanglement measure should be invariant under local unitary transformation, and its expectation should not increase under the local operation and classical communication (LOCC). First, it should be mentioned that for bipartite systems, where equation (22) gives concurrence of the state, the above conditions are satisfied. This follows from the fact that equation (22) is (up to a whole factor) the concurrence which is introduced by Albeverio *et al* in terms of local invariants [13]. On the other hand, it is also equal to the two-level concurrence defined in [14]. In [14], Fan *et al* have defined a hierarchy of concurrence for bipartite pure states and have shown that they are entanglement monotones, i.e., cannot increase under LOCC. Now, what can we say about monotonicity of the concurrence for multipartite states? For any state we define a concurrence vector the norm of which quantifies entanglement of the state. The definition is such that for the multipartite case, the concurrence vector can be regarded as a direct sum of concurrence subvectors (equation (30)) in such a way that the norm of each subvetor denotes pairwise entanglement shared between particles. However, it should be stressed that this pairwise entanglement (equation (31)) is not obtained through tracing out the remaining subsystems, but its definition comes directly from a generalization of the bipartite case. The definition is simple and computationally straightforward, and allows the concurrence to be read off from the state. Our conjecture is that such a defined measure for multipartite systems is non-increasing under LOCC; however, its proof is still in progress.

Next, to demonstrate the nature of this measure, we give some multi-qubit examples in the following.

First, let us consider a three-qubit system with the GHZ state: $|\mathrm{GHZ}_3\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$. For this state equations (31) and (33) give $C^{\{ij\}} = 1/\sqrt{2}$ (for all i, j), and $C = \sqrt{3/2}$ respectively.

As another example, let us consider the W-state and anti-W-state defined with $|W_3\rangle = (|110\rangle + |101\rangle + |011\rangle)/\sqrt{3}$ and $|\widetilde{W}_3\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$, respectively. For these states we obtain $C^{\{ij\}} = 2/3$ (for all i, j), and $C = 2/\sqrt{3}$.

For $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \otimes (\alpha_3|0\rangle + \beta_3|1\rangle)$ with $|\alpha_3|^2 + |\beta_3|^2 = 1$ we obtain $C^{\{12\}} = 1$, $C^{\{13\}} = C^{\{23\}} = 0$ and $C = 1$. The above examples show that $C(|\mathrm{GHZ}_3\rangle)C(|W_3\rangle) = C(|\widetilde{W}_3\rangle)C(|\mathrm{EPR}_{12}\rangle \otimes |\psi_3\rangle)$. Actually, in an m-qubit system, the generalized GHZ state $|\mathrm{GHZ}_m\rangle = (|0^{\otimes m}\rangle + |1^{\otimes m}\rangle)/\sqrt{2}$ has maximum total concurrence equal to $C(|\mathrm{GHZ}_m\rangle) = \sqrt{m(m-1)/2}$.

Let us now consider two superposition states which are considered in [25]. First consider a superposition of the W-state and anti-W-state, i.e., $|W\widetilde{W}(s,\phi)\rangle \equiv \sqrt{s}|W\rangle + \sqrt{1-s}\,\mathrm{e}^{\mathrm{i}\phi}|\widetilde{W}\rangle$. For this state, we obtain $C^{\{ij\}} = \frac{2}{3}\sqrt{\frac{3}{2}s(s-1)+1}$ and $C = \sqrt{3}C^{\{ij\}}$. This result is in complete agreement with the entanglement that was obtained in [25] by using a geometric measure. Consider now a superposition of the W-state and the GHZ state, i.e., $|GW(s,\phi)\rangle \equiv \sqrt{s}|\mathrm{GHZ}\rangle + \sqrt{1-s}\,\mathrm{e}^{\mathrm{i}\phi}|W\rangle$. In this case, we obtain $C^{\{ij\}} = \frac{1}{3\sqrt{2}}\sqrt{s(5s-4)+8}$ and $C = \sqrt{3}C^{\{ij\}}$. For this state, the geometric measure of [25] observes an entanglement which is dependent on phase ϕ, but our measure shows an entanglement independent of phase ϕ. However, in this example, as far as s is concerned, the behaviour of our entanglement measure is closely related to the behaviour of the geometric measure of [25].

4. Conclusion

In summary, we gave the definition of a concurrence vector and proposed to use its norm as a measure of entanglement. In the case of a bipartite pure state, it is shown that the norm of the concurrence vector leads to the other proposals of generalization of concurrence. In the multipartite case, the concurrence vector is regarded as the direct sum of concurrence subvectors, each one is associated with a pair of particles, therefore, the norm of each subvector is used as the entanglement contribution of the corresponding pair. We argue that the definition is not exhaustive in order to completely quantify entanglement, so the result of the paper is a small step towards quantifying the entanglement. Also the definition of concurrence vectors considered in this paper is just for pure states, and the problem of mixed states remains open.

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